PARALLEL POSTULATE PROJECT

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MATH 221: FOUNDATIONS OF GEOMETRY

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ABSTRACT. The aim of this paper is to cover Carl Friedrich Gauss' life, mathematical works, and most importantly his contributions to the resolution of Euclid's parallel postulate.

1. Biography

FIGURE 1. Carl Friedrich Gauss (1777 - 1855)

Johann Carl Friedrich Gauss was a German algebraist, geometry enthusiast, number theorist, and physical scientist. He is often known as "Princeps mathematicorum" (Latin, "the Prince of Mathematicians") as well as "the greatest mathematician since antiquity". Gauss was born on April 30, 1777 in Brunswick, Germany, to poor, hard working, and middle class parents. His father, Gebhard Dietrich Gauss, worked multiple jobs as a gardner, brick-layer, butcher, sales assistant, and treasurer for a local company. Though Carl's father was regarded as an upright and honest man, he was a harsh father who did not support the idea of his son going to school, with expectations that Carl would follow one of the family occupations. Carl's mother, Dorothea Benze Gauss, was considered a very intelligent woman, but unfortunately illiterate, since she did not receive education and her main job before marriage was a housemaid.

Carl's exceptional talent for numbers was shown way back when he was a little child, as he could calculate before he learned to speak. At just the age of three, he was able to find errors in his father payroll calculations, and he was checking his dad's accounts on a daily basis by the age of five. The most well-known story of Gauss' gifted math ability is a tale from when he was an elementary student. Carl amazed his arithmetic teacher with how fast he determined the sum of the first 100 positive integers. He quickly recognized that there are 50 pairs of numbers, each one of which adds up to 101, and he got 5050 by taking the product of the number of pairs and the sum of each pair. [Figure 2]

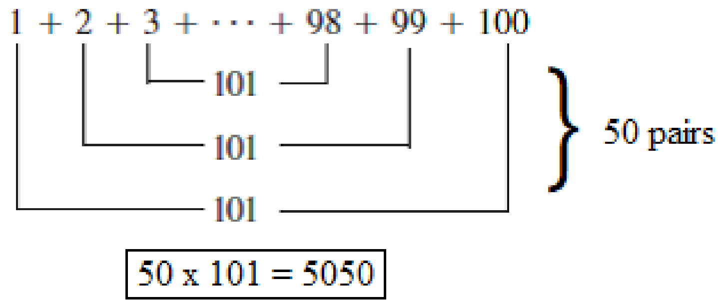

FIGURE 2. Gauss' Famous Arithmetic Problem

Gauss' math prodigy quickly received the attention of the Duke of Brunswick, who decided to provide financial support to Gauss and sent him to Brunswick Collegium Carolinum at the age of fifteen. The Duke then continued to sponsor Gauss as the young mathematician furthered his studies at the University of Gottingen. Two of Gauss' biggest accomplishments (which will be discussed in the next section) happened after college, during his 20s: proving the Fundamental Theorem of Algebra at

22 and publishing Disquisitiones Arithmeticae at 24. During his career, Gauss also had strong interest in studying astronomy, statistics, complex numbers, geometry, and many more areas of mathematics.

Later on in his life, Gauss was appointed as a professor of mathematics and also the director of the observatory at his alma mater, University of Gottingen. He worked in this official position until the day he passed away - February 23, 1855.

2. Non-Parallel postulate works

Gauss, without a doubt, is one of the most versatile mathematicians of all time. He is in the discussion for the title "greatest mathematician of all time", with other excellent candidates such as Leonard Euler, Isaac Newton, and Archimedes. Gauss made contributions to almost every area of mathematics that existed during his time, from pure mathematics fields such as number theory, algebra, analysis and geometry, to applied math topics such as statistics and probability theory, astronomy and magnetism. We are going to take a look at some of Gauss' greatest mathematical discoveries and accomplishments.

In 1976, at the age of 19, Gauss achieved his first mathematical triumph by successfully figuring out how to construct a heptadecagon - a 17-sided polygon [Figure 3], with the use of a straight edge and a compass.

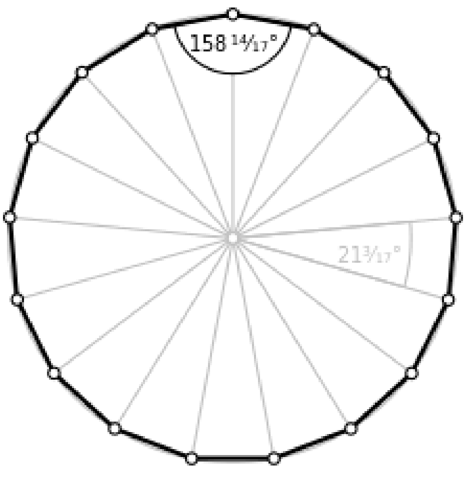

Figure 3. The Heptadecagon

Gauss' proof was based on two main ideas, expressing the trigonometric functions of the common angle in terms of arithmetic operations and square root extractions, and the odd prime factors of the number of sides of an n-gon are distinct Fermat primes, which have the form $F_n = 2^{2^n} + 1$ for some $n \in \mathbb{Z}^+$. In order to make a regular heptadecagon, it requires the value of $\cos(\frac{2\pi}{17})$ in terms of square roots, which involves a degree 17 (a Fermat prime) equation. [Figure 4]

$$\cos\left(\frac{2\pi}{17}\right) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{2\cdot 17 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{2\cdot 17 - 2\sqrt{17}} - 2\sqrt{2\cdot 17 + 2\sqrt{17}}}$$

FIGURE 4

Shortly after graduated from Gottingen, at the age of 22, Gauss proved what is now known as the Fundamental Theorem of Algebra. The theorem states that

> "Every non-constant single-variable polynomial with complex coefficients has at least one complex root. This includes polynomials with real coefficients, since every real number can be considered a complex number with its imaginary part equal to zero."

At the age of 24, Gauss published the greatest book of his career, Disquisitiones Arithmeticae. It had a great impact on the field of number theory, which Gauss once called "the queen of mathematics", as it paved the path for modern study of the integers and integer-valued functions. The book consists of seven sections, which are (I) Congruent Numbers in General; (II) Congruences of the First Degree; (III) Residues of Powers; (IV) Congruences of the Second Degree; (V) Forms and Indeterminate Equations of the Second Degree; (VI) Various Applications of the Preceding Discussions; and (VII) Equations Defining Sections of a Circle. The 7 sections are subdivided into 366 numbered items - theorems with proof, or author's remarks and thoughts.

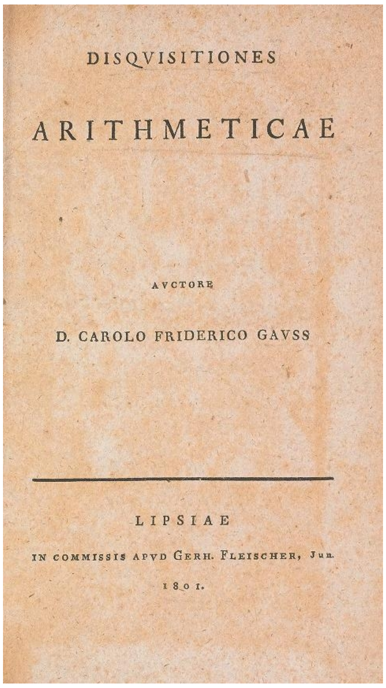

Figure 5. Disquisitiones Arithmeticae

In addition to pure mathematics, Gauss also contributed to the area of probability and statistics. Gauss introduced what is now known as Gaussian (normal) distribution, the Gaussian function and the Gaussian error curve. He showed how probability could be represented by a bell-shaped or "normal-ish" curve. The highest point on the curve represents the most probable event in a series of data (i.e. the mean or expected value), while all other possible occurrences are equally distributed around the mean value, creating a downward-sloping curve on each side of the peak.

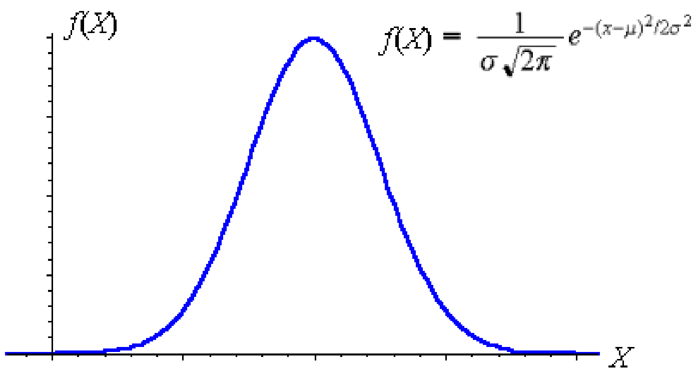

Figure 6. Normal distribution curve and density function

3. Parallel postulate contributions

3.1. A brief overview of the Parallel Postulate

Euclidean Geometry is the intuitive geometry that we naturally think about the world. However, there also exists a few other not as widely known, but about as important geometries that also have many applications in the world and the universe. These other geometries are different from our regular geometry due to the nature of parallel lines. Euclid's fifth postulate states that:

> "If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

Postulate V is also commonly called the Parallel Postulate because it is equivalent to the following statement:

> "For every line l and for every point P that does not lie on l, there is exactly one line m such that P lies on m and m is parallel to l."

Unlike the first four postulates, -

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. All right angles are equal to one another.

- it is very obvious to mathematicians, including Euclid, that there is something odd and unclear about this fifth postulate. In his greatest mathematics book Elements, Euclid actually avoided using the fifth postulate until Proposition 29. The first four were simple statements that few would feel uncertain about. On the other hand, the fifth postulate was far from being directly self-evident and somewhat difficult to interpret.

"If a straight line falling on two straight lines..."

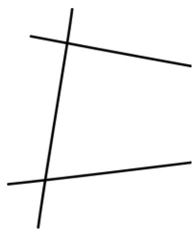

Figure 7

"...make the interior angles on the same side less than two right angles,..."

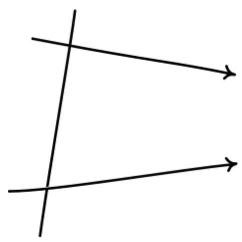

Figure 8

"...the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

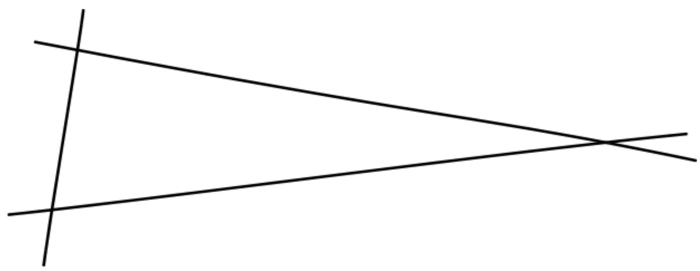

Figure 9

For decades and centuries, a number of geometry devotees have attempted to show that Postulate V is the consequence of the first four postulates. However, it turns out that this postulate determines whether we are in Euclidean or non-Euclidean geometry. Euclidean geometry is the study of geometry that satisfies all of Euclid's axioms, whereas a geometry where the parallel postulate does not hold is called non-Euclidean. Some of the main forerunners of non-Euclidean geometry include Giovanni Gerolamo Saccheri, Johann Heinrich Lambert, Adrien-Marie Legendrre, Janos Bolyai, Nikolai Ivanovich Lobachevsky, and of course, our very own Carl Friedrich Gauss.

3.2. **Gauss and the Parallel Postulate**

It is a mistake to start this section by not mentioning that it was Carl Friedrich Gauss who invented the term "Non-Euclidean Geometry." He devoted three decades of his life to the studies of parallels theory. His interest in studying this problem began from the very beginning of the 19th century.

One year before the start of the 1800s, in a letter to Wolgang (Farkas) Bolyai, Gauss mentioned that he had attempted to work on the Parallel Postulate problem:

> “I am sorry that I didn’t use our former close proximity to learn more about your work on the first principles of geometry; I would surely have spared myself considerable wasted effort and have become more tranquil, insofar as this is possible for someone like me when there is so much to be desired in this [geometry] situation. I myself have moved far ahead in my work on this (considering that my other heterogeneous tasks leave little time); the path that I have hammered out does not so much lead to the goal that one hopes for, and which you have secured, but much more it makes the truth of geometry dubious. To be sure, I have found much that would qualify as a proof for most [that Euclidean geometry is correct], but which in my eyes really proves nothing; for example, if one could prove that a straightedged triangle exists whose area would be greater than that of a given region, then I would be in the position to rigorously justify the whole of geometry. Most people would accept the former as an axiom; not me; it could be possible that no matter how far apart the vertices of a triangle are assumed to be, still the area would remain under a given bound, however far apart the three angular points of the triangle were taken. I have several such results, but in none of them can I find anything satisfactory.”

In 1824, Gauss wrote a letter to his friend Franz Adolf Taurinus, in which he mentioned that

> "The assumption that the angle sum [of a triangle] is less than 180 degree leads to a curious geometry, quite different from ours but thoroughly consistent, which I have developed to my entire satisfaction. The theorems of this geometry appear to be paradoxical, and, to the uninitiated, absurd, but calm, steady reflection reveals that they contain nothing at all impossible."

Even though Gauss is deeply interested in the topic of parallels, he never published anything. He only discussed very little detail of his work with his trusted friends, as shown by the following excerpt from a letter of Gauss to Heinrich Schumacher in 1831:

> "In the last few weeks I have begun to put down a few of my Meditations [on parallels] which are already to some extent nearly years old. These I had never put in writing, so that I have been compelled three or four times to go over the whole matter afresh in my head. Also I wished that it should not perish with me."

Gauss kept his clear view of a geometry that is independent from Postulate V a secret for almost half of a century, and it was only revealed after Nikolai Lobachevsky and Janos Bolyai announced their works on the same topic. In a letter to Janos Bolyai's father, Wolfgang Bolyai, Gauss stated that he has studied the same parallels problem for a very long time and provided enough evidence to show that he actually had worked out all the proofs and other details:

> "If I start by saying I cannot praise it then you will most likely be taken aback; but I cannot do otherwise; to praise it would be to praise myself; the entire contents of the work, the path that your son has taken and the results to which it leads, are almost perfectly in agreement with my own meditations, some going back 30 - 35 years. In truth I am astonished. My intention was not to release any of my own work in my lifetime. Most people don't have a true sense of what is involved, and I have found very few who are particularly interested. To appreciate

what is going on one must first of all have a real grasp of what is missing, and on this point most are in the dark. On the other hand it was my intention to write everything down so that it did not perish with me."

Gauss' definition of two parrallel lines is

> "If the coplanar straight lines AM and BN do not intersect each other, while, on the other hand, every straight line through A between AM and AB cuts BN, then AM is said to be parallel to BN." [Figure 10]

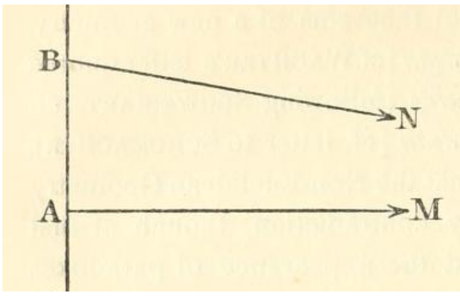

FIGURE 10

According to Gauss, there are two types of lines that begin from A and extending to the right: those that intersect BN and those that do not. If we construct these lines by extending AB upwards and rotating it around A clockwise, then the first line that does not intersect BN is parallel to BN. His reason was there can only be a unique position separating the lines which intersect BN from those that do not intersect it. This must be the first of the lines, which do not cut BN, and by definition, it is the parallel AM, since there cannot be no last line of the set of lines which intersect BN.

Gauss then went on to prove that "the parallelism of the line AP to the line BQ is independent of the points A and B, provided the sense in which the lines are to be produced indefinitely remain the same." He used three cases to prove this result:

- Case 1: If A is fixed and let $B' \in BN$ $(B' \neq B)$, then we get the same parallel AM.

- Case 2: [Figure 11] Let $A' \in AM$ where $A * A' * M$ and $A' \neq A$, and construct a line $A'P$ that lies between $A'B$ and $A'M$. Then let $Q \in A'P$ where $A' * Q * P$, and construct line AQ. By definition, AQ and BN must intersect, thus this also means QP and BN must intersect. Thus the line that A, A', M lie on is the first of the lines that do not intersect BN, and so $A'M \mathbin{/\!/} BN$.

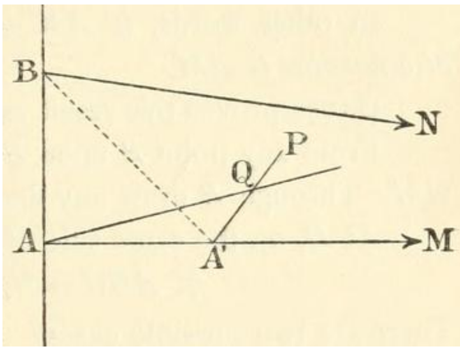

FIGURE 11

- Case 3: [Figure 12] If $A' * A * M$, we also construct a line $A'P$ that lies between $A'B$ and $A'M$. Let $Q \in A'P$ such that $P * A' * Q$. Then QA and BN must intersect by definition. Let $QA \cap BN = R$. Thus $A'P$ belongs to the interior of the closed figure $A'ARB$, and it must intersect one of the sides $A'A, AR, RB$, and BA'. Clearly that one side must be RB, and thus $A'M \mathbin{/\!/} BN$.

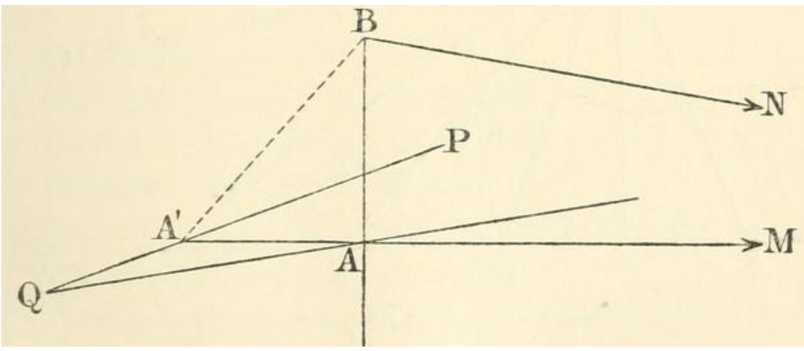

FIGURE 12

Another result that Gauss proved is the Reciprocity of Parallelism, which stated that "If $AM \mathbin{/\!/} BN$, then $BN \mathbin{/\!/} AM$." To prove this, we first drop a perpendicular from B to AM, and call the foot A. Construct a line BN' between BA and BN. On the same side of AB as BN', construct $\angle ABC$ such that $\mu(\angle ABC) = \frac{1}{2}\mu(\angle N'BN)$. Then we have two cases to consider:

- Case 1: BC and AM intersect. [Figure 13]

  Let $BC \cap AM = D$. Let $E \in DA$ such that $AE = AD$ (and D*A*E), and construct $\angle BDF \cong \angle BED$. Then DF and BM must interest, say at G, since $AM \mathbin{/\!/} BN$. Let $H \in EM$ such that $E * H * M$ and $EH = DG$. Then it is obvious that $\triangle BAD \cong \triangle BAE$ by SAS, thus $BE \cong BD$. Now $\triangle BEH \cong \triangle BDG$ by SAS, and it follows that $\angle EBH \cong \angle DBG$, or $\angle EBD \cong \angle HBG$. But $\angle EBD \cong \angle N'BN$, and this means that BN' and BH coincide, which also means $N' = H$. Thus BN' and AM must intersect, and since BN' is a line through B between BA and BN, we have $BN \mathbin{/\!/} AM$, by Gauss' parallels definition.

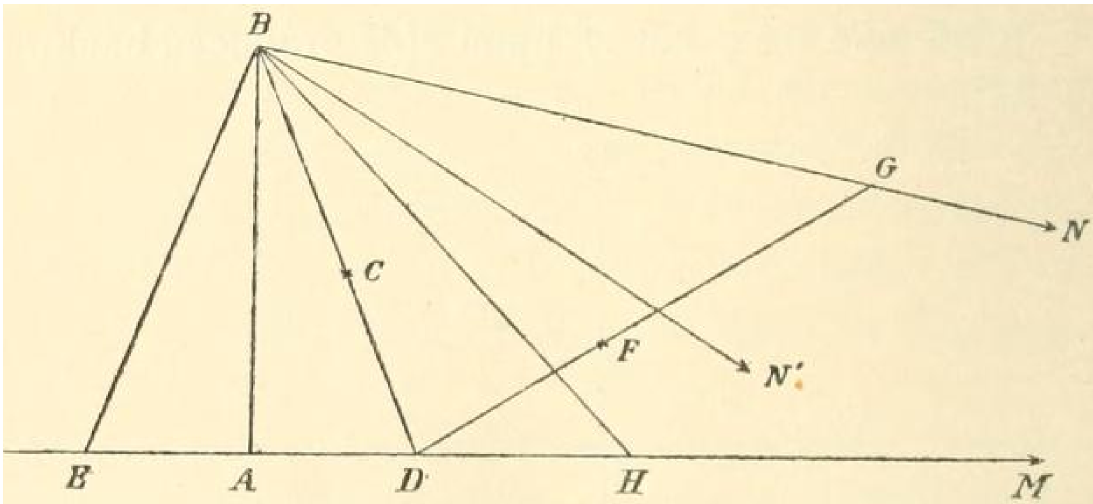

FIGURE 13

- Case 2: BC and AM do not intersect. [Figure 14]

  Let $D \in AM$. Using the same argument as the previous case, we have $\angle EBD \cong \angle GBH$. Since $\mu(\angle ABD) < \mu(\angle ABC)$, we get $\mu(\angle EBD) < \mu(\angle N'BN)$, and thus $\mu(\angle GBH) < \mu(\angle N'BN)$. Therefore BN' and AM must intersect. Since BN' is a line through B between BA and BN, we have $BN \mathbin{/\!/} AM$, by Gauss' parallels definition.

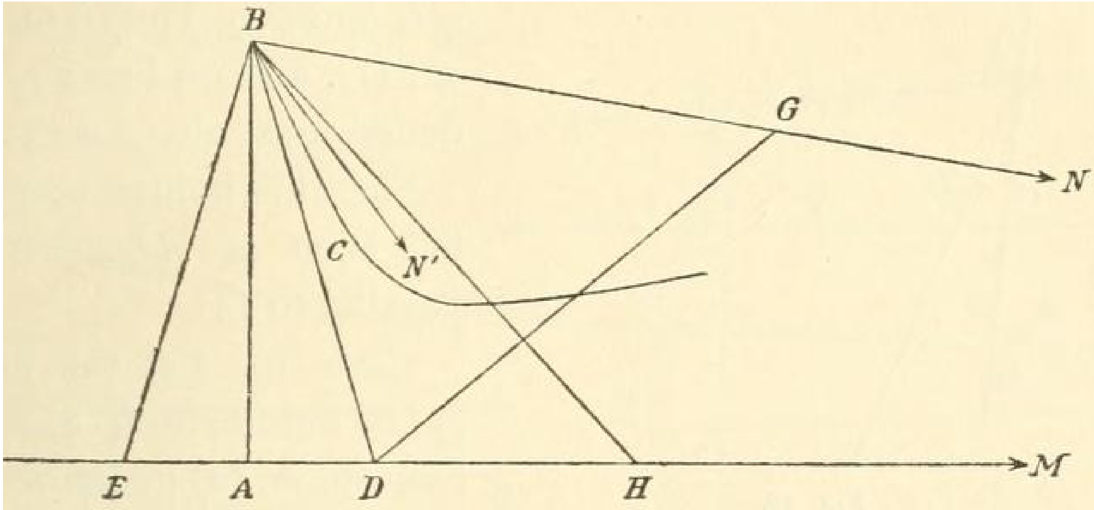

FIGURE 14

Hence in both cases we have shown that if $AM \mathbin{/\!/} BN$, then $BN \mathbin{/\!/} AM$.

Gauss also showed the transitivity of parallels, which is that "If line (1) // line (2) and line (1) // line (3), then line (2) // line (3). This proof is divided into 2 cases:

- Case 1: (1) lies between (2) and (3) [Figure 15]

  Let $A \in (2)$, $B \in (3)$, and $AB \cup (1) = C$. Construct a line AD between AB and (2). Then AD must intersect (1), and it would also intersect (3) if we extend it. Thus (2) // (3)

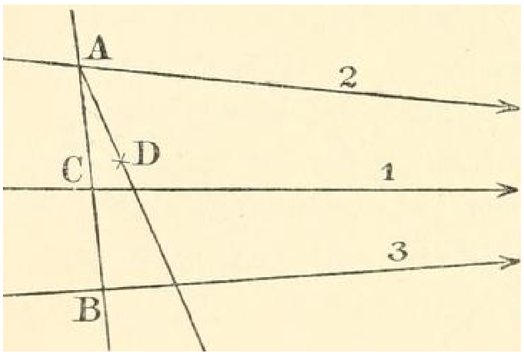

FIGURE 15

- Case 2: One of (2), (3) lies between (1) and the other [Figure 16]

  Suppose (2) lies between (1) and (3) and (2) is not parallel to (3). Then for any point on (3), we can construct a line (3') different from (3) that is parallel to (2). By the first case, (3') // (1), which is impossible. Thus (2) // (3)

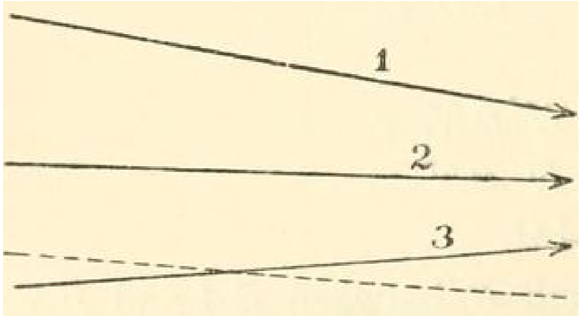

FIGURE 16

Another important parallel ideas Gauss investigated in was corresponding points on two parallel lines, which is defined as "two points A and B are said to correspond when AB makes equal internal angles with the parallels on the same side." [Figure 17]

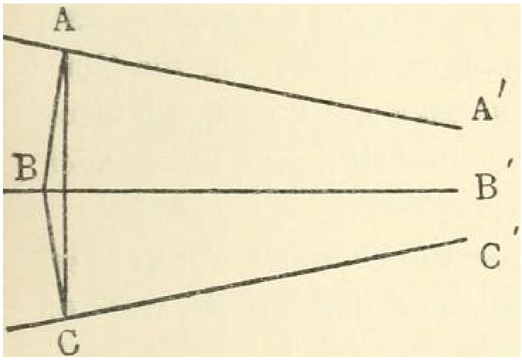

FIGURE 17

He also came up with the following theorems regarding corresponding points:

1. If A and B are two corresponding points upon two parallels, and M is the midpoint of AB, then the line MN, perpendicular to AB, is parallel to the two given lines, and every point on the same side of MN has a closer distance to A than B.

2. If A and B are two corresponding points upon the parallels l and m, and A', B' are two other corresponding points on the same lines, then $AA' = BB'$, and conversely

3. If A, B, C are three points on the parallels l, m, and n, such that A and B, B and C, correspond, then A and C also correspond.

David Hilbert once claimed that "The most suggestive and notable achievement of the last [19th] century is the discovery of non-Euclidean geometry." It would be difficult to overestimate the importance of Gauss' contributions to the study of this topic, as his significant results showed the existence of more than one consistent geometry. Gauss was a key player of the first period in the history of non-Euclidean geometry, and his and other mathematicians' works in this era had paved the way for more studies and discoveries in the second non-Euclidean geometry period, with major names like Bernhard Riemann, Hermann von Helmholtz, Sophus Lie, and Eugenio Beltrami.

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"Mathematics is the queen of the sciences - and number theory is the queen of mathematics."